Policy iteration for stochastic zero-sum games

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Hamilton-Jacobi-Bellman-Isaacs equations

The stationary equation:

$$-H(x, Dv(x)) = 0, x \in X \subset \mathbb{R}^d \quad + \text{a boundary condition},$$

$$H(x,p) = \min_{a \in \mathcal{A}(x)} \max_{b \in B(x)} [f(x,a,b) \cdot p + g(x,a,b)], \; x \in X, \text{ and } p \in \mathbb{R}^d \quad ,$$

is the dynamic programming equation satisfied by the (upper) value function of the zero-sum game problem:

$$v(x) = \inf_{(\alpha_t)_{t\geq 0}} \sup_{(\beta_t)_{t\geq 0}} \int_0^\infty g(x_t, \alpha_t, \beta_t) dt \quad ,$$

where $\dot{x}_t = f(x_t, \alpha_t, \beta_t)$, for all $t \geq 0$, and inf and sup are taken over nonanticipating strategies of the first and second player (where the second player knows the current action of the first player).

Example: pursuit evasion games.

Discretization with a monotone scheme (for instance a Kushner scheme)

$\Rightarrow$

$v = F(v)$, the fixed point equation of the dynamic programming or Shapley operator F of a discrete time zero-sum two player stochastic game problem with finite state space.

Same for: Discounted problems, Optimal stopping time problems, Stochastic games.

Discrete time and state zero-sum stochastic games

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined by:

$$[F(v)]_i := \min_{a \in \mathcal{A}_i} \max_{b \in \mathcal{B}_i} \left(\sum_{j \in [n]} M_{ij}^{ab} v_j + r_i^{ab} \right), \quad i \in [n],$$

with $M_{ij}^{ab} \geq 0$ for all $i, j \in [n], a \in \mathcal{A}_i,\ b \in \mathcal{B}_i$.

The map F is the *dynamic programming or Shapley operator* of a discrete time zero-sum two player game problem with perfect information on the finite state space $\mathcal{X} := [n] := \{1, \dots, n\}$, with:
$\mathcal{A}_i, \mathcal{B}_i$ sets of actions of the 1st, 2nd player MIN, MAX, when in state i
r_i^{ab} reward paid by MIN to MAX, at each time
$M_{ij}^{ab} := \gamma_i^{ab} P_{ij}^{ab} \geq 0$
$\gamma_i^{ab} := \sum_{j \in [n]} M_{ij}^{ab} \geq 0$ discount factor (< 1 or ≤ 1 or $= 1$)
P_{ij}^{ab} transition probability from i to j ($\sum_{j \in [n]} P_{ij}^{ab} = 1$).

Discrete time and state zero-sum stochastic games

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined by:

$$[F(v)]_i := \min_{a \in \mathcal{A}_i} \max_{b \in \mathcal{B}_i} \left(\sum_{j \in [n]} M_{ij}^{ab} v_j + r_i^{ab} \right), \quad i \in [n],$$

with $M_{ij}^{ab} \geq 0$ for all $i, j \in [n], a \in \mathcal{A}_i,\ b \in \mathcal{B}_i$.
Denote $\gamma_i^{ab} := \sum_{j \in [n]} M_{ij}^{ab}$.

Then

- F is order preserving: $u \leq v \Rightarrow F(u) \leq F(v)$, for all $u, v \in \mathbb{R}^n$;
- if $\gamma_i^{ab} \leq 1$ for all $i \in [n]$ and $a \in \mathcal{A},\ b \in \mathcal{B}$, then F is additively sub-homogeneous: $F(\lambda + u) \leq \lambda + F(u)$, for all $\lambda \geq 0$ and $u \in \mathbb{R}^n$
- thus F is sup-norm nonexpansive.
- If $\gamma_i^{ab} = 1$ for all $i \in [n]$ and $a \in \mathcal{A},\ b \in \mathcal{B}$, then F is additively homogeneous: $F(\lambda + u) = \lambda + F(u)$, for all $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^n$.

Let the value function of the game with infinite horizon be given by:

$$v_x = \inf_{(\alpha_k)_{k\geq 0}} \sup_{(\beta_k)_{k\geq 0}} \mathbb{E}\left[\sum_{k=0}^{\infty}(\prod_{\ell=0}^{k-1}\gamma_{X_\ell}^{\alpha_\ell,\beta_\ell}) r_{X_k}^{\alpha_k,\beta_k} \mid X_0 = x\right] \quad ,$$

where α_k and β_k are possible strategies of both players of the game (at time k), and $X_k \in [n]$ is the state process of the game satisfying $P(X_{k+1} = j | X_k = i, \alpha_k = a, \beta_k = b) = P_{ij}^{ab}$.

If $\gamma_x^{a,b} \leq \bar{\gamma} < 1$, then F is a sup-norm contraction:

$$\|F(v) - F(w)\|_\infty \leq \bar{\gamma}\|v - w\|_\infty \quad ,$$

and v is the unique solution of

$$v = F(v).$$

Moreover the optimal actions in $F(v)$ give the optimal stationary strategies of the game.

Solving stationary dynamic programming equations

Problem: compute $v \in \mathbb{R}^n$ such that $F(v) = v$, when such a solution is unique, and bound the complexity of this computation.

When $\gamma_i^{ab} \leq \bar{\gamma} < 1$ for all $i \in [n]$ and $a \in \mathcal{A},\ b \in \mathcal{B}$, then

- Then, the *value iterations* coincide with fixed point iterations: $v^{k+1} = F(v^k)$, and with the finite horizon approximations with $T = k$ and $\varphi = v^0$. They converge geometrically towards v with factor $\bar{\gamma}$:

$$\lim_{k \to \infty} \|v^k - v\|^{1/k} \leq \bar{\gamma} \ .$$

- However, the value iteration algorithm is only pseudopolynomial.
- Also the existence of a polynomial algorithm is an open problem.
- What about the policy iteration?

Policy iterations for discounted games

Assume: $\mathcal{A}_i$ and $\mathcal{B}_i$ are finite sets, and
Denote by $\Sigma := \{\sigma : i \in [n] \mapsto \sigma_i \in \mathcal{A}_i\}$ and $\Delta := \{\delta : i \in [n] \mapsto \delta_i \in \mathcal{B}_i\}$ the sets of policies,
and for $\sigma \in \Sigma$ and $\delta \in \Delta$, define the matrices and vectors:

$$M^{(\sigma\delta)} = (M_{ij}^{\sigma_i\delta_i})_{ij=1,\dots,n}, \quad \text{and } r^{(\sigma\delta)} = (r_i^{\sigma_i\delta_i})_{i=1,\dots,n} \ ,$$

and the affine maps

$$F^{(\sigma\delta)}(v) = M^{(\sigma\delta)}v + r^{(\sigma\delta)}, \quad v \in \mathbb{R}^n \ .$$

Then, F can be written as:

$$F(v) = \min_{\sigma\in\Sigma} F^{(\sigma)}(v) \ , \text{ with } F^{(\sigma)}(v) := \max_{\delta\in\Delta} F^{(\sigma\delta)}(v) \ , \quad v \in \mathbb{R}^n \ ,$$

where minima and maxima are for the partial order of $\mathbb{R}^n$.
The maps $F^{(\sigma\delta)}$, $F^{(\sigma)}$ and F are all order preserving and contracting for the sup-norm with contraction factor $\bar{\gamma}$.
Important: the infimum and supremum are attained because the sets $\{F^{(\sigma)}(v) \mid \sigma \in \Sigma\}$ and $\{F^{(\sigma\delta)}(v) \mid \delta \in \Delta\}$ are rectangular.

Policy iterations for discounted games

(Howard, 1960) for 1-player games, (Denardo, 1967) for 2-player games.

Using operators:

Given an initial policy $\sigma^0 \in \Sigma$, apply successively the two following steps for $s \geq 0$ until $\sigma^{s+1} = \sigma^s$:

1. Compute the fixed point v^s of $F^{(\sigma^s)}$;
2. Improve the policy: choose an optimal policy for v^s, that is $\sigma^{s+1} \in \Sigma$ such that $F(v^s) = F^{(\sigma^{s+1})}(v^s)$ with $\sigma^{s+1} = \sigma^s$ as soon as this is possible.

Step 1 is solved by using Policy iteration for the (one-player) game with fixed policy σ^s, which constructs $v^{s,l}$ and $\delta^{s,l}$ from $\delta^{s,0}$.

Policy iterations for discounted games

(Howard, 1960) for 1-player games, (Denardo, 1967) for 2-player games.

With control terminology:

Given an initial policy $\sigma^0 \in \Sigma$, apply successively the two following steps for $s \geq 0$ until $\sigma^{s+1} = \sigma^s$:

1. Compute the value v^s of the game with fixed policy σ^s, that is the solution of $v = F^{(\sigma^s)}(v)$;
2. Improve the policy: choose an optimal policy for v^s, that is $\sigma^{s+1} \in \Sigma$ such that $F(v^s) = F^{(\sigma^{s+1})}(v^s)$ or equivalently:

$$\sigma_i^{s+1} \in \operatorname*{argmin}_{a \in \mathcal{A}_i} \left\{ \max_{b \in \mathcal{B}_i} \left(\sum_{j \in [n]} M_{ij}^{ab} \, v_j^s + r_i^{ab} \right) \right\} , \quad i \in [n],$$

with $\sigma^{s+1} = \sigma^s$ as soon as this is possible.

Policy iterations for discounted games

(Howard, 1960) for 1-player games, (Denardo, 1967) for 2-player games.

Simplex algorithm for 1-player games with Dantzig pivoting:

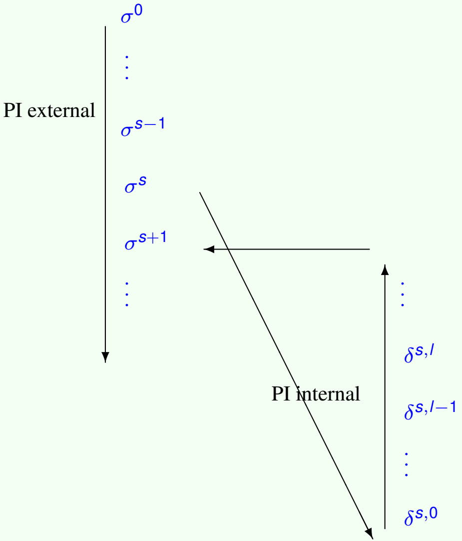
Given an initial policy $\sigma^0 \in \Sigma$, apply successively the two following steps for $s \geq 0$ until $\sigma^{s+1} = \sigma^s$:

1. Compute the value v^s of the game with fixed policy σ^s, that is the solution of $v = F^{(\sigma^s)}(v)$;
2. Improve the policy: choose a policy $\sigma^{s+1} \in \Sigma$ such that

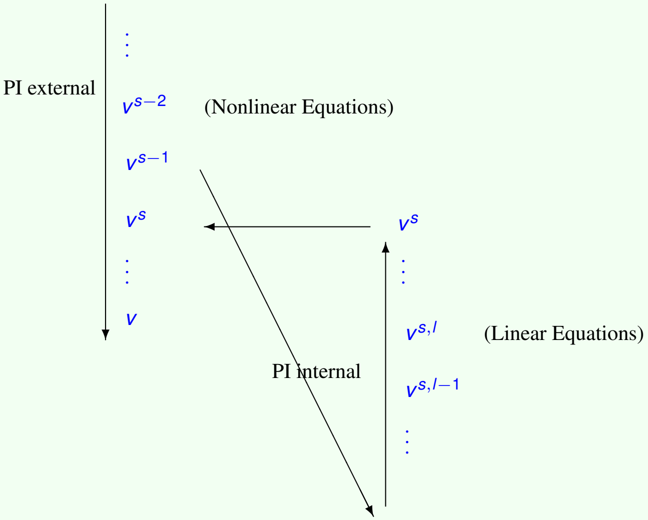
$$\sigma_i^{s+1} \in \operatorname*{argmin}_{a \in \mathcal{A}_i} \left\{ \max_{b \in \mathcal{B}_i} \left(\sum_{j \in [n]} M_{ij}^{ab} v_j^s + r_i^{ab} \right) \right\}, \quad i \in [n],$$

for one i such that $(F^{(\sigma^s)}(v^s) - F(v^s))_i$ is maximal.

Policy iterations for discounted games



Policy iterations for discounted games



Policy iterations for discounted games: monotone convergence

- The sequence $(v^s)_{s\geq 0}$ is nonincreasing;
- Hence, the sequence $(\sigma^s)_{s\geq 0}$ does not visit the same policy two times, until it becomes stationary;
- So the sequence $(v^s)_s$ is stationary after a finite time (at most $\sharp\Sigma$), and converges towards the solution v of $v = F(v)$.

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- When s is fixed, the sequence $(v^{s,l})_l$ is nondecreasing;
- Hence, the sequence $(\delta^{s,l})_l$ does not visit the same policy two times, until it becomes stationary;
- So the sequence $(v^{s,l})_l$ is stationary after a finite time (at most $\sharp\Delta$), and converges towards the solution v^s of $v = F^{(\sigma^s)}(v)$.

Policy iterations for discounted games: well known properties

- The Policy iterations converge faster than the value iterations: for all $s \geq 0$, $v \leq v^{s+1} \leq F(v^s) \leq v^s$, so $v \leq v^s \leq F^s(v^0) \leq v^0$.
- If the discount factor is uniformely bounded by some constant $\bar{\gamma} < 1$, then for the sup-norm, we have:

$$\|v^{s+1} - v\| \leq \|F(v^s) - v\| \leq \bar{\gamma}\|v^s - v\| \ .$$

- For 1-player games with an infinite number of actions and under regularity conditions, Policy iterations coincide with the Newton algorithm, and have a *super-linear convergence*.

- However, in general, the number of (external) iterations is bounded by $\sharp\Sigma \geq 2^n$ if $\sharp\mathcal{A}_i \geq 2$ for all $i \in [n]$.

Policy iterations for discounted games: recent results

- (Friedmann, 2009) showed a 2-player deterministic game problem with $\gamma \simeq 1$ and an exponential number of iterations.
- (Fearnley, 2010) and (Andersson, 2009) showed the same for a 1-player stochastic game.

Policy iterations for discounted games: recent results

(Ye, 2011) showed that Policy iteration algorithm and Simplex algorithm solve 1-player discounted games with fixed discount factor $\gamma < 1$ in *strongly polynomial* time.
(Hansen, Miltersen and Zwick, 2011) extended and improved this result to Policy iteration algorithm for 2-player games. They show that the number of iterations $s_{\max}$ (to obtain stationarity) satisfies:

$$s_{\max} \leq (m+1)\left(1 + \frac{\log(n^2/(1-\gamma))}{-\log(\gamma)}\right) = \mathcal{O}\left(\frac{m}{1-\gamma} \log \frac{n}{1-\gamma}\right),$$

with $m =$ the *total number of actions*: the number of (i, a, b) with $i \in [n]$, $a \in \mathcal{A}_i$ and $b \in \mathcal{B}_i$.
(Feinberg, Huang, 2013): Same for a one-player game with mean-payoff, and a state i_0 such that $P^a_{i,i_0} \geq 1 - \gamma$, for all $i \in [n],\ a \in \mathcal{A}_i$.

Question: What remains true when the discount factors γ_i^{ab} are not uniformely bounded by a constant < 1?
or for games with mean-payoff?

Theorem (A., Gaubert, arXiv:1310.4953)

Let us fix $0 < \lambda < 1$. The policy iteration algorithm for the class of 2-player games satisfying

$$r(M^{(\sigma\delta)}) \leq \lambda \quad \forall \sigma \in \Sigma,\ \delta \in \Delta$$

is strongly polynomial. More precisely, the number of external iterations $s_{\max}$ satisfies:

$$s_{\max} \leq (m_1 - n)(1 + \lfloor \frac{\log(1-\lambda)}{\log(\lambda)} \rfloor) = \mathcal{O}(\frac{m_1 - n}{1-\lambda} \log \frac{1}{1-\lambda}),$$

with $m_1 =$ the total number of actions of the first player: the number of (i, a) with $i \in [n]$ and $a \in \mathcal{A}_i$.

Proof.

- Adapt the proof of (Hansen, Miltersen and Zwick, 2011) by using sup-norms instead of ℓ_1 norms and the nonlinear maps $F^{(\delta)}$ to obtain the above bound when the discount factors are $\leq \lambda$. A similar bound is obtained by (Scherrer, 2013) in the one-player case with fixed discount factor.
- Using nonlinear spectral theory, show that for all $\lambda < \mu < 1$, there exists $\varphi \in \mathbb{R}^n$ such that $\varphi_i > 0$, $i \in [n]$, and $M^{(\sigma\delta)}\varphi \leq \mu\varphi$.
- Let $G(v) = \varphi^{-1}F(\varphi v)$ with $\varphi v = (\varphi_i v_i)_{i\in[n]}$. Then G is the dynamic programming operator of a game with discount factors $\leq \mu$, and the sequence of policies σ^s for F and G are the same, so is $s_{\max}$.
- Equivalently, F is contracting on $\mathbb{R}^n$ with contraction factor μ, for the weighted sup-norm $\|\cdot\|_\varphi$ defined by:

$$\|v\|_\varphi := \max_{i\in[n]} \left|\frac{v_i}{\varphi_i}\right| \quad \forall v \in \mathbb{R}^n \ .$$

- Take the infimum of the bound over all μ.

$\square$

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- Let $G(v) = \varphi^{-1}F(\varphi v)$ with $\varphi v = (\varphi_i v_i)_{i\in[n]}$. Then G is the dynamic programming operator of a game with discount factors $\leq \mu$, and the sequence of policies σ^s for F and G are the same, so is $s_{\max}$.
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- Take the infimum of the bound over all μ.

$\square$

Definition (Nonlinear spectral radii (Nussbaum,Mallet-Paret, 1998))

Let h be a nonlinear continuous positively homogenous map on a closed convex cone C of $\mathbb{R}^n$ ($h(\lambda v) = \lambda h(v)$ for all $\lambda > 0$ and $v \in C$):

- The *cone eigenvalue spectral radius* of h, $\hat{r}_C(h)$, is the maximal modulus of an eigenvalue of h in C, where λ is an eigenvalue associated to $v \in C\backslash\{0\}$ if $h(v) = \lambda v$.
- The *Collatz-Wielandt number* $\mathrm{cw}_C(h)$ is the infimum of the super-eigenvalues of h, where $\lambda > 0$ is a super-eigenvalue if there exists v in the interior of C such that $h(v) \leq \lambda v$.
- The *Bonsall's spectral radius* of h is defined as:

$$r_C(h) := \inf_{k\geq 1} \|h^k\|_C^{1/k}, \quad \text{with} \quad \|h\|_C := \sup_{x\in C,\ \|x\|=1} \|h(x)\| \ ,$$

for any given norm $\|\cdot\|$ on $\mathbb{R}^n$.

Theorem (Nussbaum, LAA 1986, also (A., Gaubert, Nussbaum, arXiv 2011))

For a continuous, positively homogenous, order preserving selfmap h of $C = \mathbb{R}_+^n$, all the above spectral radius notions of h coincide:

$$\begin{aligned} r(h) &= \inf_{k\geq 1} \|h^k\|_{\mathbb{R}_+^n}^{1/k} \\ &= \max\{\lambda \in \mathbb{R} \mid \exists v \in \mathbb{R}_+^n \setminus \{0\}, h(v) = \lambda v\} \\ &= \inf\{\lambda > 0 \mid \exists v \in (\mathbb{R}_+^*)^n,\ h(v) \leq \lambda v\} \end{aligned}$$

Proposition (A. Gaubert, Nussbaum, arXiv 2011)

Assume that h and h_π are continuous, positively homogenous, order preserving selfmaps of $\mathbb{R}_+^n$, for all $\pi \in \Pi$, and that $h(v) = \max_{\pi\in\Pi} h_\pi(v)$ for all $v \in \mathbb{R}_+^n$, then

$$r(h) = \max_{\pi\in\Pi} r(h_\pi)\ .$$

Applying the proposition to $h(v) := \max_{\sigma\in\Sigma} \max_{\delta\in\Delta}(M^{(\sigma\delta)}v)$, we get that $r(h) \leq \lambda < \mu$ and so by the theorem, there exists $\varphi \in (\mathbb{R}_+^*)^n$ such that $M^{(\sigma\delta)}\varphi \leq h(\varphi) \leq \mu\varphi$, for all $\sigma \in \Sigma,\ \delta \in \Delta$.

Consider the value function of the game with mean-payoff:

$$\eta_x = \inf_{(\alpha_k)_{k\geq 0}} \sup_{(\beta_k)_{k\geq 0}} \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{k=0}^{T-1} r_{x_k}^{\alpha_k,\beta_k} \mid X_0 = x\right] .$$

Let F be the dynamic programming operator such that $\gamma_i^{ab} \equiv 1$.
F is additively homogeneous. We say that $v \in \mathbb{R}^n$ is an *(nonlinear additive) eigenvector* or *biais* of F with *eigenvalue* $\rho \in \mathbb{R}$ if $F(v) = \rho + v$.

- If ρ exists, then $\eta_x = \rho$ for all $x \in [n]$.
- If all the matrices $M^{(\sigma\delta)}$ are irreducible, then ρ exists and the eigenvector v is unique up to an additive constant.

- Other existence results of ρ: Bather, 1973, Gaubert, Gunawardena, 2001.

Policy iterations for "irreducible" mean-payoff games

(Hoffman and Karp, 1966) We have to solve $\rho + v = F(v)$.

Using operators:

Given an initial policy $\sigma^0 \in \Sigma$, apply successively the two following steps for $s \geq 0$ until $\sigma^{s+1} = \sigma^s$:

1. Compute the additive eigenvalue and eigenvector ρ^s and v^s of $F^{(\sigma^s)}$, that is the solution of $\rho + v = F^{(\sigma^s)}(v)$;
2. Improve the policy: choose an optimal policy for v^s, that is $\sigma^{s+1} \in \Sigma$ such that $F(v^s) = F^{(\sigma^{s+1})}(v^s)$ with $\sigma^{s+1} = \sigma^s$ as soon as this is possible.

Step 1 is solved by using Policy iteration for the (one-player) game with fixed policy σ^s, which constructs $\rho^{s,l}$, $v^{s,l}$ and $\delta^{s,l}$ from $\delta^{s,0}$.

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With control terminology:

Given an initial policy $\sigma^0 \in \Sigma$, apply successively the two following steps for $s \geq 0$ until $\sigma^{s+1} = \sigma^s$:

1. Compute the value ρ^s and the biais v^s of the game with fixed policy σ^s, that is the solution of $\rho + v = F^{(\sigma^s)}(v)$;
2. Improve the policy: choose an optimal policy for v^s, that is $\sigma^{s+1} \in \Sigma$ such that $F(v^s) = F^{(\sigma^{s+1})}(v^s)$ or equivalently:

$$\sigma_i^{s+1} \in \operatorname*{argmin}_{a \in \mathcal{A}} \left\{ \max_{b \in \mathcal{B}} \left(\sum_{j \in [n]} M_{ij}^{ab} v_j^s + r_i^{ab} \right) \right\}, \quad i \in [n],$$

with $\sigma^{s+1} = \sigma^s$ as soon as this is possible.

Policy iterations for "irreducible" mean-payoff games: monotone convergence

- The sequence $(\rho^s)_{s\geq 0}$ is nonincreasing;
- If $\rho^s = \rho^{s+1}$, then $v^s - v^{s+1}$ is constant and $v^s = v$.
- Hence, the sequence $(\sigma^s)_{s\geq 0}$ does not visit the same policy two times, until it becomes stationary;
- So the sequence $(\rho^s, v^s)_s$ is stationary after a finite time (at most $\sharp\Sigma$), up to an additive constant, and converges towards the solution (ρ, v) of $\rho + v = F(v)$.

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- When s is fixed, the sequence $(\rho^{s,l})_l$ is nondecreasing;
- If $\rho^{s,l} = \rho^{s,l+1}$, then $v^{s,l} - v^{s,l+1}$ is constant and $v^{s,l} = v^s$.
- Hence, the sequence $(\delta^{s,l})_l$ does not visit the same policy two times, until it becomes stationary;
- So the sequence $(\rho^{s,l}, v^{s,l})_l$ is stationary after a finite time (at most $\sharp\Delta$), and converges towards the solution (ρ^s, v^s) of $\rho + v = F^{(\sigma^s)}(v)$.

For a Markov matrix M and states i, j, denote:

$$\mathcal{T}_{ij}(M) = \mathbb{E}[\inf\{k \geq 1 \mid X_k = j\} \mid X_0 = i] \ ,$$

the expected first return (or hitting) time in state j, starting from i. Note that $\mathcal{T}_{ii_0}(M) < +\infty$ for all $i \in [n]$ if and only if M has a unique recurrent (final) class and i_0 belongs to it.

Theorem (A., Gaubert, arXiv:1310.4953)

Let us fix $K > 0$ and a state i_0. The policy iteration algorithm for the class of 2-player mean-payoff games such that

$$\mathcal{T}_{ii_0}(M^{(\sigma\delta)}) \leq K \quad \forall \sigma \in \Sigma,\ \delta \in \Delta,\ i \in [n]$$

is strongly polynomial. More precisely, the number of external iterations $s_{\max}$ satisfies:

$$s_{\max} \leq (m_1 - n)(1 + \lfloor \frac{\log(K)}{\log(K/(K-1))} \rfloor) = \mathcal{O}((m_1 - n)K \log K),$$

with $m_1 =$ the total number of actions of the first player.

Sketch of the proof. • Let $\varphi \in (\mathbb{R}_+^*)^n$ be defined by: $\varphi_i = \max_{\sigma\in\Sigma}\max_{\delta\in\Delta}\mathcal{T}_{ii_0}(M^{(\sigma\delta)})$.

- Let $Q^{(\sigma\delta)}$ be obtained from $M^{(\sigma\delta)}$ by putting its i_0th column to zero. Then $\varphi = 1 + \max_{\sigma\in\Sigma}\max_{\delta\in\Delta}(Q^{(\sigma\delta)}\varphi)$.
- Let $N^{(\sigma\delta)}$ be obtained from $M^{(\sigma\delta)}$ by replacing its i_0th column by the nonnegative vector $(\varphi - 1 - Q^{(\sigma\delta)}\varphi)/\varphi_{i_0}$.
- $N^{(\sigma\delta)}$ has nonnegative entries and satisfies:

$$N^{(\sigma\delta)}\varphi = \varphi - 1 \leq \lambda\varphi \quad \text{with } \lambda = 1 - 1/K \;\Rightarrow\; r(N^{(\sigma\delta)}) \leq \lambda \;.$$

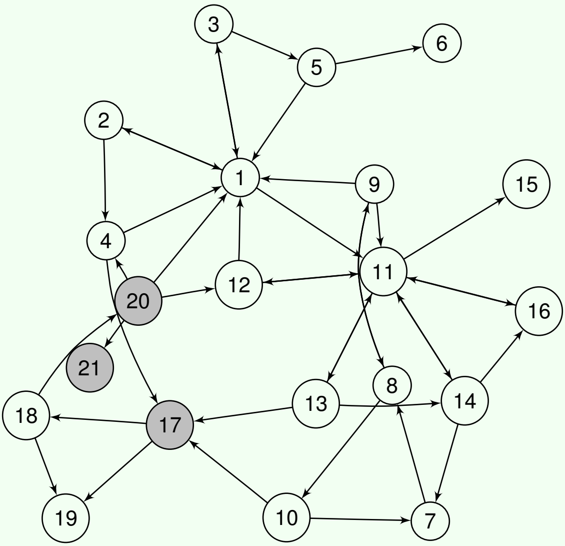
- Then the map

$$G(v) = \min_{\sigma\in\Sigma}\max_{\delta\in\Delta}(N^{(\sigma\delta)}v + r^{(\sigma\delta)}) \;, \quad v \in \mathbb{R}^n$$

  satisfies the assumptions of the theorem for discounted games.
- If $v_{i_0} = 0$, then $\rho + v = F(v) \Leftrightarrow \rho\varphi + v = G(\rho\varphi + v)$.
- Hence, the sequences of policies σ^s and $\delta^{s,l}$ for F and G are the same.

□

Example: Spammer vs. Web search engine



Nodes = web pages
Arcs = hyperlinks
21 : spammer page
1 : non controlled page.

Associated Markov matrix S: $S_{ij} = 1/N_i$ if (i, j) is an hyperlink, $S_{ij} = 0$ otherwise; $N_i =$ number of hyperlinks from i.

The PageRank is the invariant measure π of S.

- Let v be the preference probability vector of the Web search engine
- Let α be a damping factor: the probability for a Web surfer to use the Web seach engine.
- Usually, one replaces S by $\alpha S + (1-\alpha)\mathbb{1}v$, $\mathbb{1} = (1 \cdots 1)^T$.
- Similar to consider the Markov matrix of the Web with the Web search engine: $M = \begin{bmatrix} 0 & v \\ \alpha\mathbb{1} & (1-\alpha)S \end{bmatrix}$.
- If r is an instantaneous reward such that $r_i = 1$ for $i = s$ and 0 otherwise, then the mean-payoff is the PageRank (frequency of visit) π_s of the spammer site s.
- Optimizing the spammer site is a 1-player game with mean-payoff (see for instance (Fercoq, A., Bouhtou, Gaubert, IEEE TAC 2013).

A zero-sum game problem:

- $\sigma \in \Sigma$ is the policy of the Web search engine, it controls v and wants to minimize the PageRank of the spammer site;
- $\delta \in \Delta$ is the policy of the spammer, it controls the rows of S with index in his site, and wants to maximize its PageRank.
- All final classes of $M^{(\sigma\delta)}$ contain state 1 (the Web search engine).

In the general case, we need to apply Policy iterations for multichain mean-payoff games,...

and to find a complexity result.

Related recent results for 1-player discounted games

- (Post, Ye, 2012) show that the simplex algorithm for deterministic MDP (1-player games) is strongly polynomial independently of the discount factor: it stops after $\mathcal{O}(n^5 m^2 \log^2 n)$ iterations, where m is the number of possible actions by state (thus $m_1 = nm$).
- (Scherrer, 2013) generalizes this result to stochastic MDP which satisfy a bound which may be seen (and is equal when the discount factor γ tends to 1) as a bound τ_r on the expected first return time to recurrent states and a bound τ_t on the expected exit time from transient states. Under these conditions the simplex algorithm stops after $\mathcal{O}(n^3 m^2 \tau_r \tau_t \log^2(n\tau_r\tau_t))$.
- (Scherrer, 2013) shows a similar result for Policy Iteration algorithm for stochastic MDP (1-player games), when the set of transient states is independent of the strategy. Under these conditions the Policy iteration algorithm stops after $n(m-1)(\lceil \tau_r \log(n\tau_r)\rceil + \lceil \tau_t \log(n\tau_t)\rceil)$ iterations.
- However, this assumption implies that the recurrent classes are independent of the strategy.

Theorem (A., Gaubert, 2014)

Let us fix $K > 0$ and a state i_0. The policy iteration algorithm for the class of 2-player discounted games with fixed discount factor, $M^{(\sigma\delta)} = \gamma P^{(\sigma\delta)}$ with $\gamma < 1$, such that

$$\mathcal{T}_{ii_0}(P^{(\sigma\delta)}) \leq K \quad \forall \sigma \in \Sigma,\ \delta \in \Delta,\ i \in [n]$$

is strongly polynomial. More precisely, the number of external iterations $s_{\max}$ satisfies:

$$s_{\max} \leq (m_1 - n)(1 + \lfloor \frac{\log(K)}{\log(K/(K-1))} \rfloor) = \mathcal{O}((m_1 - n)K \log K),$$

with $m_1 =$ the total number of actions of the first player.
Hence the bound does not depend on γ.

For a Markov matrix M, a state i and set C of states, denote:

$$\mathcal{T}_{iC}(M) = \mathbb{E}[\inf\{k \geq 1 \mid X_k \in C\} \mid X_0 = i] \ ,$$

the expected first return (or hitting) time in set C, starting from i.

Theorem (A., Gaubert, 2014)

Let us fix $K > 0$ and a subset C of states with cardinality s. The policy iteration algorithm for the class of 2-player multichain mean-payoff games such that for all $\sigma \in \Sigma, \ \delta \in \Delta$, each final class of $M^{(\sigma\delta)}$ contains exactly one element of C and

$$\mathcal{T}_{iC}(M^{(\sigma\delta)}) \leq K \quad \forall i \in [n]$$

is strongly polynomial. More precisely, the number of external iterations $s_{\max}$ satisfies:

$$s_{\max} \leq (m_1 - n)(1 + \lfloor \frac{\log(sK)}{\log(sK/(sK-1))} \rfloor) = \mathcal{O}((m_1 - n)sK\log(sK)),$$

with $m_1 =$ the total number of actions of the first player.

Multichain mean-payoff games

- In general, F may not have additive eigenvalue and eigenvector, that is ρ and v such that $\rho + v = F(v)$.
- If the action spaces $\mathcal{A}_i$ and $\mathcal{B}_i$ are finite for all $i \in [n]$, then F is *polyhedral*, and since it is also nonexpansive, by the Kohlberg (1980) theorem, there exist η and v in $\mathbb{R}^n$ such that

$$F(t\eta + v) = (t+1)\eta + v, \text{ for } t \text{ large enough.}$$

- (η, v) is called an *invariant half-line*.
- Then η is the value of the game with mean-payoff.
- Moreover, there exist $\hat{F}$ and $\acute{F}_{\cdot}$ such that (η, v) is an invariant half-line if and only if it satisfies the system:

$$\begin{cases} \eta &= \hat{F}(\eta) \;, \\ \eta + v &= \acute{F}_{\eta}(v) \;. \end{cases}$$

- However v is not unique.

Policy iterations for multichain mean-payoff games

Construct a sequence of policies σ^s, values η^s and biais v^s.
They were introduced and proved to converge by

- (Howard, 1960) and (Denardo and Fox, 1968) for 1-player multichain mean-payoff games,
- (Vöge and Jurdziński, 2000) for parity games,
- (Cochet-Terrasson, Gaubert, Gunawardena, 1998 and 1999), (Bjorklund, Sandberg, Vorobyov, 2004), (Jurdziński,Paterson, Zwick, 2006) for 2-player deterministic games,
- (Cochet-Terrasson and Gaubert, 2006), (A., Cochet-Terrasson, Detournay, and Gaubert, arXiv:1208.0446, and CDC 2013), (Detournay,PIGAMES library, 2012), (Bourque, Raghavan, preprint, 2012) for general multichain 2-player stochastic games. (Detournay, 2012).

To avoid cycling, one need to add some constraints on v^s, for instance:

- fix the value $v_i^s = 0$ at one point i of each final class of $M^{(\sigma\delta)}$ (Howard, and Denardo and Fox, for one-player games);
- by a nonlinear projection (Cochet-Terrasson and Gaubert);

and to choose optimal policies in a conservative way.

Summary:

- The policy iteration algorithm for discounted games is strongly polynomial when restricted to the class of games such that *the spectral radii of all $M^{(\sigma\delta)}$ are bounded by $\lambda < 1$*. This result is invariant by diagonal scaling.
- The policy iteration algorithm for ergodic mean-payoff games is strongly polynomial when restricted to the class of ergodic games such that *the expected first return (or hitting) time in some fixed state i_0 of the Markov chain associated to any $M^{(\sigma\delta)}$ and initial state is bounded by $K < \infty$*.
- Same result for *discounted games*.
- Same result for multichain mean-payoff games, when i_0 is replaced by a set of states C, and each recurrence class contains exactly one element of C.

Open:

- Is the policy iteration algorithm for multichain stochastic games strongly polynomial, under some more general constraints on the $M^{(\sigma\delta)}$ (only)?